International Journal of Theoretical Physics, Vol. 39, No. 3, 2000

The Description of Joint Quantum Entities and the Formulation of a Paradox[†]

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Received December 8, 1999

We formulate a paradox in relation to the description of a joint entity consisting of two subentities by standard quantum mechanics. We put forward a proposal for a possible solution, entailing the interpretation of 'density states' as 'pure states.' We explain where the inspiration for this proposal comes from and how its validity can be tested experimentally. We discuss the consequences of the proposal for quantum axiomatics.

1. FORMULATION OF THE PARADOX

Quantum mechanics, after more than 70 years, still poses fundamental problems of understanding. Many physicists believe these problems are 'only' problems of 'physical interpretation' of the mathematically well defined 'standard formalism.' In this paper we will show that this is not necessarily so. We will show that the problem of quantum mechanics connected to the existence of nonproduct states in the description of the joint entity of two quantum entities may well indicate that a change of the standard formalism is necessary.

By the 'standard formalism' of quantum mechanics we mean the formalism as exposed, for example, in von Neumann (1932), and we will refer to it by SQM. Often the name 'pure state' is used to indicate a state represented by a ray of the Hilbert space. We will use it, however, in the physical sense: a 'pure state' of an entity represents the reality of this entity. As a consequence it is natural to demand that an entity 'exists' if and only if at any moment

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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0020-7748/00/0300-0485$18.00/0

it is in one and only one 'pure state.' Let us formulate this as a principle, since it will play a major role in our analysis.

Physical Principle 1. If we consider an entity, then this entity 'exists' at a certain moment iff it 'is' in one and only one pure state at that moment.

We denote pure states of an entity S by means of symbols $p, q, r, \ldots$ and the set of all pure states by Σ. We mention that in Aerts (1984a, 1999a), where aspects of the problem that we investigate in the present paper are analyzed, a 'pure state' is called a 'state.'

A state represented by a ray of the Hilbert space will be called a 'ray state.' We denote rays by symbols $\bar{x}, \bar{y}, \ldots$, where $x, y, \ldots \in \mathcal{H}$, and we denote by p_x the 'ray state' represented by the ray $\bar{x}$. To each ray $\bar{x}$, $x \in \mathcal{H}$, corresponds one and only one ray state p_x. One of the principles of standard quantum mechanics is that 'pure states' are 'ray states.'

SQM Principle 1. Consider an entity S described by SQM in a Hilbert space $\mathcal{H}$. Each ray state p_x, $x \in \mathcal{H}$, is a pure state of S, and each pure state of S is of this form.

The problem that we want to consider in this paper appears in the SQM description of the joint entity S of two quantum entities S_1 and S_2.

SQM Principle 2. If we consider two quantum entities S_1 and S_2 described by SQM in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then the joint quantum entity S consisting of these two quantum entities is described by SQM in the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. The subentities S_1 and S_2 are in ray states p_{x_1} and p_{x_2}, with $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$, iff the joint entity S is in a ray state $p_{x_1 \otimes x_2}$.

In relation to the situation of a joint entity consisting of two subentities, there is another physical principle we generally imagine to be satisfied.

Physical Principle 2. If an entity is the joint entity of two subentities, then the entity exists at a certain moment iff the subentities exist at that moment.

Let us introduce the concept of 'nonproduct vectors' of the tensor product. For $z \in \mathcal{H}_1 \otimes \mathcal{H}_2$ we say that z is a nonproduct vector iff $\nexists\, z_1 \in \mathcal{H}_1$, $z_2 \in \mathcal{H}_2$: $z = z_1 \otimes z_2$. We are now ready to formulate the theorem that points out the paradox we want to bring forward.

Theorem 1. Physical Principle 1, Physical Principle 2, SQM Principle 1, and SQM Principle 2 cannot be satisfied together.

Proof. Suppose the four principles are satisfied. This leads to a contradiction. Consider the joint entity S of two subentities S_1 and S_2 described by SQM. From SQM Principle 2 it follows that if S_1 and S_2 are described in

Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then S is described in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let us consider a nonproduct vector $z \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and the ray state p_z. From SQM Principle 1 it follows that p_z represents a pure state of S. Consider a moment where S is in state p_z. Physical Principle 1 implies that S exists at that moment and from Physical Principle 2 we infer that S_1 and S_2 also exist at that moment. Physical Principle 1 then implies that S_1 and S_2 are respectively in pure states p_1 and p_2. From SQM Principle 1 it follows that there are two rays $\bar{z}_1$ and $\bar{z}_2$, $z_1 \in \mathcal{H}_1$ and $z_2 \in \mathcal{H}_2$, such that $p_1 = p_{z_1}$ and $p_2 = p_{z_2}$. From SQM Principle 2 it then follows that S is in the state $p_{z_1 \otimes z_2}$, which is not p_z since the ray generated by $z_1 \otimes z_2$ is different from $\bar{z}$. Since both p_z and $p_{z_1 \otimes z_2}$ are pure states, this contradicts Physical Principle 1.

2. AN ALTERNATIVE SOLUTION TO THE PARADOX

The fundamental problems of the SQM description of the joint entity of two subentities had already been remarked a long time ago. The Einstein–Podolsky–Rosen paradox and later research on the Bell inequalities are related to this difficulty (Einstein *et al.*, 1935; Bell, 1964). It is indeed states like p_z, with z a nonproduct vector, that give rise to the violation of the Bell inequalities and that generate the typical EPR correlations between the subentities. Most of the attention in this earlier analysis went to the 'nonlocal' character of these EPR correlations. The states of type p_z are now generally called 'entangled' states. The problem (paradox) related to entangled states as outlined in Section 1 has often been overlooked, although noticed and partly mentioned in some texts (e.g., Van Fraassen, 1991, Section 7.3, and references therein).

The problem of the description of a joint entity has also been studied within axiomatic approaches to SQM. There it was shown that some of the axioms that are needed for SQM are not satisfied for certain well-defined situations of a joint entity consisting of two subentities (Aerts, 1982, 1984a; Pulmannová, 1983, 1985; Randall and Foulis, 1981). More specifically, it has been shown in Aerts (1982) that the joint entity of two separated entities cannot be described by SQM because of two axioms: weak modularity and the covering law. This shortcoming of SQM is proven to be at the origin of the EPR paradox (Aerts, 1984b, 1985a, b). It has also been shown that different formulations of the product within the mathematical categories employed in the axiomatic structures do not coincide with the tensor product of Hilbert spaces (Aerts, 1984a; Pulmannová, 1983, 1985; Randall and Foulis, 1981). Again certain axioms, orthocomplementation, covering law, and atomicity, cause problems.

All these findings indicate that we are confronted with a deep problem that has several complicated and subtle aspects. A very extreme attitude would be to consider entangled states as artifacts of the mathematical formalism and hence not really existing in nature. Yet, entangled states are readily prepared in the laboratory and the corresponding EPR correlations have been detected in a convincing way. This means that it is very plausible to acknowledge the existence of entangled states as 'pure states' of the joint entity in the sense of Physical Principle 1.

As a result of earlier research we have always been inclined to believe that we should drop Physical Principle 2 to resolve the paradox (Aerts, 1984a, 1985a, b; see also Aerts *et al.*, 1999b). This entails considering the subentities S_1 and S_2 of the joint entity S as 'not existing' if the joint entity is in an entangled state. We still believe that this is a possible solution to the paradox and refer, for example, to Valckenborgh (n.d.) and Aerts and Valckenborgh (n.d.) for further structural elaboration in this direction. In the present paper we would like to bring forward an alternative solution. To make it explicit we introduce the concept of 'density state,' which is a state represented by a density operator of the Hilbert space. We denote density operators by symbols $W, V, \ldots$ and the corresponding density states by $p_W, p_V, \ldots$. To each density operator W on $\mathscr{H}$ corresponds one and only one density state p_W. Within SQM, density states are supposed to represent mixtures, i.e., situations of lack of knowledge about the pure state. The way out of the paradox we propose in the present paper consists in considering the density states as pure states. Hence, in thin sense, SQM Principle 1 would be false and replaced by a new principle.

CQM Principle 1. Consider an entity S described in a Hilbert space $\mathscr{H}$. Each density state p_W, where W is a density operator of $\mathscr{H}$, is a pure state of S, and each pure state of S is of this form.

We call the quantum mechanics that retains all the old principles except SQM Principle 1, and that follows our new principle CQM Principle 1, 'completed quantum mechanics' and refer to it by CQM.

The first argument for our proposal of this solution comes from earlier work in relation to the violation of Bell inequalities by means of macroscopic entities (Aerts, 1991a). There we introduced a macroscopic material entity that entails EPR correlations. Let us briefly describe this entity again to state our point.

First we represent the spin of a spin-1/2 quantum entity by means of the elastic sphere model that we have used on several occasions (Aerts, 1986, 1987, 1991a, b, 1993, 1995, 1999a, b), and that we have called the 'quantum machine.' It is well known that the states, ray states as well as density states, of the spin of a spin-1/2 entity can be represented by the points of a sphere B in three-dimensional Euclidean space with radius 1 and center 0. Let us denote the state corresponding to the point $w \in B$ by p_w. To make the representation explicit we remark that each vector $w \in B$ can uniquely be written as a convex linear combination of two vectors $v = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $-v$ on the surface of the sphere (Fig. 2), i.e., $w = a \cdot v - b \cdot v = (a - b) \cdot v$, with $a, b \in [0,1]$ and $a + b = 1$. In this way we correspond with w the density operator $W(w)$:

$$W(w) = \begin{pmatrix} a\cos^2\frac{\theta}{2} + b\sin^2\frac{\theta}{2} & (a-b)\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\phi} \\ (a-b)\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi} & a\sin^2\frac{\theta}{2} + b\cos^2\frac{\theta}{2} \end{pmatrix} \quad (1)$$

Each density operator can be written in this form and hence the inverse correspondence is also made explicit. We remark that the ray states, namely the density operators that are projections, correspond to the points on the surface of B.

It is much less known that experiments on the spin of a spin-1/2 quantum entity can be represented within the same picture. Let us denote the direction, in which the spin will be measured by the diametrically opposed vectors u and $-u$ of the surface of B (Fig. 1), and let us consider u as the z direction of the standard spin representation (this does not restrict the generality of

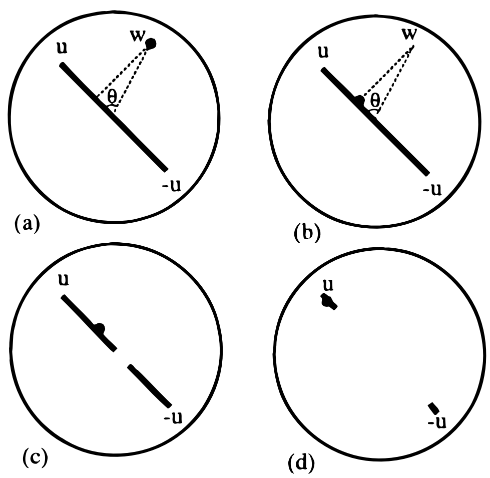

Fig. 1. A representation of the quantum machine. (a) The particle is in state p_w, and the elastic corresponding to the experiment e_u is installed between the two diametrically opposed points u and $-u$. (b) The particle falls orthogonally onto the elastic and sticks to it. (c) The elastic breaks and the particle is pulled toward the point u, such that (d) it arrives at the point u, and the experiment e_u gets the outcome "up."

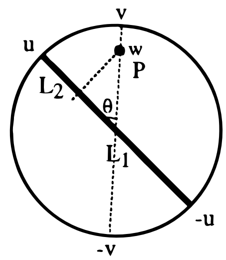

Fig. 2. A representation of the experimental process. An elastic of length 2, corresponding to the experiment e_u, is installed between u and $-u$. The probability $\mu(e_u, p_w, \text{up})$ that the particle ends at point u under influence of the experiment e_u is given by the length of the piece of elastic L_1 divided by the total length of the elastic. The probability $\mu(e_u, p_w, \text{down})$ that the particle ends at point $-u$ is given by the length of the piece of elastic L_2 divided by the total length of the elastic.

our calculation). In this case, in SQM, the spin measurement along u, which we denote e_u, is represented by the self-adjoint operator $S = \frac{1}{2}E_1 - \frac{1}{2}E_2$ with

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

being the spectral projections. The SQM transition probabilities, $\mu(e_u, p_w, \text{up})$, the probability for spin-up outcome if the state is p_w, and $\mu(e_u, p_w, \text{down})$, the probability for spin-down outcome if the state is p_w, are then

$$\mu(e_u, p_w, \text{up}) = \text{tr}(W(w) \cdot E_1) = a \cos^2 \frac{\theta}{2} + b \sin^2 \frac{\theta}{2} \tag{3}$$

$$\mu(e_u, p_w, \text{down}) = \text{tr}(W(w) \cdot E_2) = a \sin^2 \frac{\theta}{2} + b \cos^2 \frac{\theta}{2}$$

Let us now show that, using the sphere picture, we can propose a realizable mechanistic procedure that gives rise to the same probabilities and can therefore represent the spin measurement. Our mechanistic procedure starts by installing an elastic strip (e.g., a rubber band) of 2 units of length such that it is fixed with one of its endpoints at u and the other endpoint at $-u$ (Fig. 1a). Once the elastic is installed, the particle falls from its original place w orthogonally onto the elastic and sticks to it (Fig. 1b). Then, the elastic breaks at some arbitrary point. Consequently, the particle, attached to one of the two pieces of the elastic (Fig. 1c), is pulled to one of the two endpoints u or $-u$ (Fig. 1d). Now, depending on whether the particle arrives at u (as in Fig. 1) or at $-u$, we give the outcome 'up' or 'down,' respectively, to this experiment e_u.

Let us prove that the transition probabilities are the same as those calculated by SQM. The probability $\mu(e_u, p_w, \text{up})$ that the particle ends up at point u (experiment e_u gives outcome 'up') is given by the length of the piece of elastic L_1 divided by the total length of the elastic. The probability $\mu(e_u, p_w, \text{down})$ that the particle ends up at point $-u$ (experiment e_u gives outcome 'down') is given by the length of the piece of elastic L_2 divided by the total length of the elastic. This means that we have

$$\mu(e_u, p_w, \text{up}) = \frac{L_1}{2} = \frac{1}{2}(1 + (a - b)\cos\theta) = a\cos^2\frac{\theta}{2} + b\sin^2\frac{\theta}{2} \quad (4)$$

$$\mu(e_u, p_w, \text{down}) = \frac{L_2}{2} = \frac{1}{2}(1 - (a - b)\cos\theta) = a\sin^2\frac{\theta}{2} + b\cos^2\frac{\theta}{2}$$

Comparing (3) and (4), we see that our mechanistic procedure represents the quantum mechanical measurement of the spin.

To realize the macroscopic model with EPR correlations we consider two such 'quantum machine' spin models where the point particles are connected by a rigid rod, which introduces the correlation. The rigid rod is fixed and can only turn around its middle point. We will only describe the situation where we realize a state that is equivalent to the singlet spin state p_s, where $s = u_1 \otimes u_2 - u_2 \otimes u_1$, and refer to Aerts (1991a) for a more detailed analysis. Suppose that the particles are in states p_{w_1} and p_{w_2}, where w_1 *and* w_2 are, respectively, the centers of the spheres B_1 and B_2 (Fig. 3) connected by a rigid rod. We call this state (the presence of the rod included) p_w. The experiment $e_{(u_1,u_2)}$ consists in performing e_{u_1} in B_1 and e_{u_2} in B_2 and collecting the outcomes (up, up), (up, down), (down, up), or (down, down). In Fig. 3 we show the different phases of the experiment. We make the hypothesis that one of the elastics breaks first and pulls one of the particles up or down. Then we also make the hypothesis that once one of the particles has reached one of the outcomes, the rigid connection breaks down. The experiment

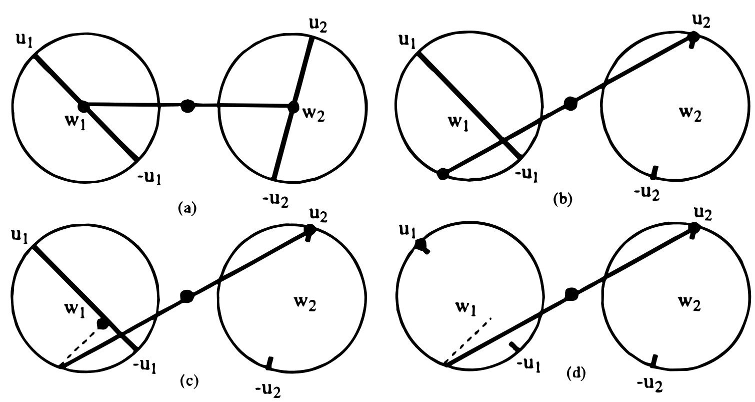

Fig. 3. A macroscopic mechanical entity with EPR correlations.

continues then without connection in the sphere where the elastic is not yet broken. The joint probabilities can now easily be calculated:

$$\begin{aligned}\mu(e_{(u_1,u_2)}, p_w, (\text{up}, \text{up})) &= \frac{1}{2}\sin^2\frac{\alpha}{2}\\ \mu(e_{(u_1,u_2)}, p_w, (\text{up}, \text{down})) &= \frac{1}{2}\cos^2\frac{\alpha}{2}\\ \mu(e_{(u_1,u_2)}, p_w, (\text{down}, \text{up})) &= \frac{1}{2}\cos^2\frac{\alpha}{2}\\ \mu(e_{(u_1,u_2)}, p_w, (\text{down}, \text{down})) &= \frac{1}{2}\sin^2\frac{\alpha}{2}\end{aligned} \tag{5}$$

where α is the angle between u_1 and u_2. These are exactly the quantum probabilities when p_w represents the singlet spin state. As a consequence, our model is a representation of the singlet spin state. This means that we can put $p_s = p_w$.

Why does this example inspire us to put forward the hypothesis that density states are pure states? Well, if we consider the singlet spin state, then this is obviously a nonproduct state, and hence the states of the subentities are density states. In fact they are the density states p_{W_1} and p_{W_2}, where

$$W_1 = W_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{6}$$

However, the state of the joint entity is clearly not given by the density state corresponding to the density operator

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{7}$$

because this state does not entail correlations. It is due to the presence of the EPR correlations that the state of the joint entity is represented by a ray state. In our macroscopic mechanistic model, however, all the states (also the states of the subentities) are ‘pure states’ and not mixtures (remark that we use the concept ‘pure state’ as defined in Section 1). If our proposal were true, namely, if density states as well as ray states in principle represented pure states, we could also understand why, although the state of the joint entity uniquely determines the states of the subentities, and hence Physical Principle 2 is satisfied, the inverse is not true: the states of the subentities do not determine the state of the joint entity. Indeed, a state of one subentity cannot contain the information about the presence of an eventual correlation between the subentities. This way, it is natural that different types of correlations can give rise to different states of the joint entity, the subentities being in the same states. This possibility expresses the philosophical principle that the whole is greater than the sum of its parts, and, as our model, shows it is also true in the macroscopic world.

Let us now say some words about the generality of the construction that inspired us for the proposed solution. It has been shown in Coecke (1995a) that a quantum-machine-like model can be realized for higher dimensional quantum entities. Coecke (1995b, 1996) also showed that all the states of the tensor product can be realized by introducing correlations on the different component states. This means that we can recover all the nonproduct ray states of the tensor product Hilbert space by identifying them with a product state plus a specific correlation for a general quantum entity, and hence that our solution of the paradox is a possible solution for a general quantum entity.

3. EXPERIMENTAL TESTING OF THE SOLUTION

If we carefully analyze the calculations that show the equivalence of our model to the quantum model, we can understand why the distinction between 'interpreting density states as mixtures' and 'interpreting density states as pure states' cannot be made experimentally. Indeed, because of the linearity of the trace used to calculate the quantum transition probabilities, and because the inner points of the sphere can be written as convex linear combinations of the surface points, an ontological situation of mixtures must give the same experimental results as an ontological situation of pure states.

If we could realize experimentally a nonlinear evolution of one of the subentities that has been brought into an entangled state with the other subentity as subentity of a joint entity, it would be possible to test our hypothesis and to detect experimentally whether density states are pure states or mixtures. Indeed, suppose that a nonlinear evolution of one of the entangled subentities could be realized. Then, we can distinguish the two possibilities in the following way. If the density state p_{W_1} of the entangled subentity is a mixture, then this state evolves while staying a convex linear combination of the ray states p_{v_1} and p_{-v_1} (referring to the situation of Fig. 3). The nonlinear evolution causes the ray states p_{v_1} and p_{-v_1} to evolve and this determines the evolution of the density state p_{W_1}, but the correspondence between p_{W_1} and p_{v_1} and p_{-v_1} remains linear. If the density state p_{W_1} of the entangled subentity is a pure state, then the nonlinear evolution will make it evolve independent of the way in which the ray states p_{v_1} and p_{-v_1} evolve. This means that in general the relation between p_{W_1} and p_{v_1} and p_{-v_1} will not remain that of a convex linear combination. So we can conclude that for a nonlinear evolution the change of the density state of an entangled subentity under this evolution will be different depending on whether it is a mixture or a pure state. This difference can be detected experimentally by a proper experimental setup. We believe that such an experiment would be of great importance for the problem that we have outlined here.

4. CONSEQUENCES FOR QUANTUM AXIOMATICS

The quantum axiomatic approaches make use of Piron's representation theorem where the set of pure states is represented as the set of rays of a generalized Hilbert space (Piron, 1964, 1976). This theorem has been elaborated and the result of Solèr has made it possible to formulate an axiomatics that characterizes SQM for real, complex, or quaternionic Hilbert spaces (Sòler 1995; Aerts and Van Steirteghem, 2000). This standard axiomatic approach aims to represent pure states by rays of the Hilbert space. If our proposal is true, an axiomatic system should be constructed that aims at representing pure states by means of density operators of the Hilbert space. Within the generalization of the Geneva–Brussels approach that we have formulated recently, and where the mathematical category is that of state property systems and their morphisms, such an axiomatic can be developed (Aerts, 1999a; Aerts *et al.* 1999a; Van Steirteghem, 2000; Van der Voorde, 2000). In Aerts (1999b) we made a small step in the direction of developing such an axiomatic system by introducing the concept of 'atomic pure states' and treating them as earlier the pure states were treated, aiming to represent these atomic pure states by the rays of a Hilbert space. We proved that in this case the covering law remains a problematic axiom in relation to the description of the joint entity of two subentities (Theorem 18 of Aerts, 1999b). We are convinced that we would gain a better understanding of the joint entity problem if a new axiomatic could be worked out aiming to represent pure states by density operators of a Hilbert space, and we are planning to engage in such a project in the coming years.

ACKNOWLEDGMENTS

The author is a Senior Research Associate of the Belgian Fund for Scientific Research. This research was carried out within the project BIL 96-03 of the Flemish Government.

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